LINKED GRASSMANNIANS AND CRUDE LIMIT LINEAR SERIES

BRIAN OSSERMAN

ABSTRACT. In [7], a new construction of limit linear series is presented which functorializes and compactifies the original construction of Eisenbud and Harris, using a new space called the linked Grassmannian. The boundary of the compactification consists of crude limit series, and maps with positive-dimensional fibers to crude limit series of Eisenbud and Harris. In this paper, we carry out a careful analysis of the linked Grassmannian to obtain an upper bound on the dimension of the fibers of the map on crude limit series, thereby concluding an upper bound on the dimension of the locus of crude limit series, and obtaining a simple proof of the Brill-Noether theorem using only the limit linear series machinery. We also see that on a general reducible curve, even crude limit series may be smoothed to nearby fibers.

1. INTRODUCTION

In [2], Eisenbud and Harris introduced the powerful theory of limit linear series. In [7], a new construction of spaces of limit linear series is introduced, which is functorial and provides a compactification of the Eisenbud-Harris space, agreeing with it on the open locus of "refined" limit series. Coincidentally, concrete motivation for such a construction has been provided by Khosla [6], who produces, given a suitable proper stack of limit linear series, an infinite family of effective virtual divisors in $\mathcal{M}_g$, and shows that whenever they are divisors, they give counter-examples to the Harris-Morrison slope conjecture.

The Eisenbud-Harris construction is forced to omit what they call the locus of "crude" limit series; they give a fiber-by-fiber description of this boundary, but do not include it in the relative construction which is the heart of the theory. The theory of [7] has a boundary which maps naturally to the Eisenbud-Harris crude limit series, but frequently with positive-dimensional fibers. Because of this distinction, we will refer to the boundary elements of the latter construction as crude limit series, and the boundary described by Eisenbud and Harris as EH-crude limit series.

Dimension estimates are central to both theories of limit linear series, and while EH-crude limit series are easily amenable to making such estimates inductively, the crude limit series of [7] are more combinatorially complicated, and were not closely analyzed in [7]. The goal of the present paper is to obtain sufficiently sharp upper bounds for the dimensions of spaces of crude limit series that we can apply the theoretical machinery of [7] to the loci of crude limit series in addition to refined limit series. Our estimates will allow us to prove the following theorem.

Theorem 1.1. *Fix integers r, d, and let X be a general curve of compact type over* $\operatorname{Spec} k$ *having no more than two components, with* $\operatorname{char} k = 0$, *and general marked points. Then the space of limit linear series on X of degree d and dimension r, with prescribed ramification at the marked points, is proper, and pure of exactly the expected dimension* $\rho = (r+1)(d-r) - rg - \sum_{i,j} \alpha_j^i$.

If no ramification is specified, this space is non-empty, and if further $\rho > 0$, the space is connected.

Here by a general curve of compact type, we mean that the dual graph and the genus of each component may be specified, and then the isomorphism class of each component must be allowed to be general. In fact, everything except the connectedness will follow easily from the theory of [7], so in particular we obtain a new and direct proof of the Brill-Noether theorem for linear series with prescribed ramification [2, Thm. 4.5] in characteristic 0. We also see that on a general curve, all limit series, including crude limit series, are smoothable to nearby fibers. We thus obtain positive answers to Questions 7.1 and 7.3 of [7].

Unfortunately, the presence of inseparable linear series poses an obstacle to carrying through the same proof in positive characteristic, although a different proof for the case of 1-dimensional linear series is given in [8]. Finally, we mention the complementary result [5, Thm. 4.3] that when a limit linear space has the expected dimension, it is Cohen-Macaulay and flat over the base. We therefore conclude that over a general curve with two components in characteristic 0, limit linear series schemes are quite well behaved.

See §5 below for background on limit linear series and smoothing families. The main theorem is proved by careful analysis of the linked Grassmannian, which arises in the limit linear series construction of [7]. We begin in §2 by reviewing the definition of and basic results on the linked Grassmannian. We then focus our attention in §3 on the map which projects to the first and last subspaces, introducing some notation and stating background lemmas to set up a more detailed analysis of the fibers of this map, which is carried out in §4. Finally, we carry out the stated application to spaces of limit linear series in §5.

2. Review of the linked Grassmannian

We briefly review the basic definitions and results of the linked Grassmannian.

Definition 2.1. Let S be an integral, locally Cohen-Macaulay scheme, and $\mathscr{E}_1, \ldots, \mathscr{E}_n$ vector bundles on S, each of rank d. Given maps $f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$ and $g_i : \mathscr{E}_{i+1} \to \mathscr{E}_i$, and a positive integer $r < d$, we denote by $\mathcal{LG} := \mathcal{LG}(r, \{\mathscr{E}_i\}_i, \{f_i, g_i\}_i)$ the functor associating to each S-scheme T the set of sub-bundles $V_1, \ldots, V_n$ of $\mathscr{E}_{1,T}, \ldots, \mathscr{E}_{n,T}$ having rank r and satisfying $f_{i,T}(V_i) \subset V_{i+1}$, $g_{i,T}(V_{i+1}) \subset V_i$ for all i.

We say that $\mathcal{LG}$ is a **linked Grassmannian** functor if the following further conditions on the f_i and g_i are satisfied:

(I) There exists some $s \in \mathscr{O}_S$ such that $f_i g_i = g_i f_i$ is scalar multiplication by s for all i.

(II) Wherever s vanishes, the kernel of f_i is precisely equal to the image of g_i, and vice versa. More precisely, for any i and given any two integers r_1 and r_2 such that $r_1 + r_2 < d$, then the closed subscheme of S obtained as the locus where f_i has rank less than or equal to r_1 and g_i has rank less than or equal to r_2 is empty.

(III) At any point of S, $\operatorname{im} f_i \cap \ker f_{i+1} = 0$, and $\operatorname{im} g_{i+1} \cap \ker g_i = 0$. More precisely, for any integer r_1, and any i, we have locally closed subschemes of S corresponding to the locus where f_i has rank exactly r_1, and $f_{i+1}f_i$ has rank less than or equal to $r_1 - 1$, and similarly for the g_i. Then we require simply that all of these subschemes be empty.

The main theorem of [7] on the linked Grassmannian is the following:

Theorem 2.2. [7, Lem. A.3, Thm. A.15] $\mathcal{LG}$ *is representable by a scheme LG; this scheme is naturally a closed subscheme of the obvious product $G_1 \times \cdots \times G_n$ of Grassmannian schemes over S, which is smooth of relative dimension $nr(d-r)$. Each component of LG has codimension $(n-1)r(d-r)$ in the product, and maps surjectively to S. If s is non-zero, then LG is also irreducible.*

We consider the following question, motivated by applications to the theory of limit linear series:

Question 2.3. Let $LG = LG(E_i, f_i, g_i)$ be a linked Grassmannian space over $\operatorname{Spec} k$, and $\operatorname{pr}_{1n} : LG \to G_1 \times G_n$ the projection map determined by forgetting all but the first and last subspaces. What are the dimensions of the fibers of this map?

We thus fix $(V_1, V_n) \in G_1 \times G_n$, and want to consider the ways of filling in the intermediate $V_i \subset E_i$ to obtain a collection of spaces linked by the f_i and g_i. Because we are motivated primarily by applications to limit linear series, we will in fact be interested primarily in obtaining upper bounds rather than computing the precise answer. Furthermore, because the problem is otherwise trivial, we fix the following assumptions:

Situation 2.4. We have fixed vector spaces E_i and maps $f_i : E_i \to E_{i+1}$, and $g_i : E_{i+1} \to E_i$ satisfying the hypotheses of Definition 2.1. We further assume that $n > 2$ and the s of condition (I) Definition 2.1 is equal to 0. Finally, we fix $V_1 \subset E_1$ and $V_n \subset E_n$ such that the iterated image of V_1 under the f_i is contained in V_n, and the iterated image of the V_n under the g_i is contained in V_1.

We remark that although the last condition is certainly necessary for the pair (V_1, V_n) to come from a point of the linked Grassmannian, it is by no means sufficient. In the course of our analysis we will produce a sufficient condition, albeit an extremely unwieldy one.

3. Notation and background lemmas

We begin with some preliminary definitions and observations.

Notation 3.1. For $1 \le i \le j \le n-1$, we denote by $f_{i,j}$ the composition $f_j \circ \cdots \circ f_i$, and $g_{j,i}$ the composition $g_i \circ \cdots \circ g_j$.

Given $V_1 \subset E_1$ and $V_n \subset E_n$, for $1 \le i \le n$, we write:

$$\bar{V}_{1,i} := g_{i-1,1}(g_{i-1,1}^{-1}(V_1)) = V_1 \cap \operatorname{im} g_{i-1,1} \subset V_1, \text{ and}$$

$$\bar{V}_{n,i} := f_{i,n-1}(f_{i,n-1}^{-1}(V_1)) = V_1 \cap \operatorname{im} f_{i,n-1} \subset V_n,$$

where by convention $g_{0,1}$ and $f_{n,n-1}$ are just the identity map, so that $\bar{V}_{1,1} = V_1$, and $\bar{V}_{n,n} = V_n$. We also write

$$V_{1,n} := g_{n-1,1}(V_n), \text{ and } V_{n,1} := f_{1,n-1}(V_1).$$

We thus obtain filtrations

$$V_{1,n} \subset \bar{V}_{1,n} \subset \bar{V}_{1,n-1} \subset \cdots \subset \bar{V}_{1,2} \subset V_1$$

and

$$V_{n,1} \subset \bar{V}_{n,1} \subset \bar{V}_{n,2} \subset \cdots \subset \bar{V}_{n,n-1} \subset V_n.$$

Note that the first containments of each filtration make use of our hypothesis that $g_{n-1,1}(V_n) \subset V_1$ and $f_{1,n-1}(V_1) \subset V_n$.

We now observe that any intermediate set of V_i linking V_1 and V_n can be constructed within the spaces $\bar{V}_{1,i} \oplus \bar{V}_{n,i}$.

Lemma 3.2. *Let $\{V_i\}$ be a point of a linked Grassmannian $LG(E_i, f_i, g_i)$. Then for each $i = 2, \ldots, n-1$ there is a natural injection*

$$V_i \hookrightarrow g_{i-1,1}^{-1}(V_1) \cap f_{i,n-1}^{-1}(V_n) \hookrightarrow \bar{V}_{1,i} \oplus \bar{V}_{n,i},$$

defined by the map $g_{i-1,1} \oplus f_{i,n-1}$.

Proof. The first inclusion inside E_i follows from the assumption that the V_i are all linked under the f_i and g_i. For the second, we only need to see that any vector in E_i mapping to 0 under $g_{i-1,1}$ and $f_{i,n-1}$ must itself be 0. By condition (III) of Definition 2.1, we have $\ker g_{i-1,1} = \ker g_{i-1}$ and $\ker f_{i,n-1} = \ker f_i$, and further $\ker f_i \cap \operatorname{im} f_{i-1} = 0$. But by condition (II) of *loc. cit.*, $\ker g_{i-1} = \operatorname{im} f_{i-1}$ and is hence disjoint from $\ker f_i$, completing the proof. □

Notation 3.3. In the situation of the lemma, we denote by $\bar{Z}_i$ the cokernel of the map $g_{i-1,1}^{-1}(V_1) \cap f_{i,n-1}^{-1}(V_n) \hookrightarrow \bar{V}_{1,i} \oplus \bar{V}_{n,i}$, and by $\bar{Z}_{1,i}$ and $\bar{Z}_{n,i}$ the images of $\bar{V}_{1,i} \oplus (0)$ and $(0) \oplus \bar{V}_{n,i}$, respectively.

We next observe the following:

Lemma 3.4. *For all $i = 2, \ldots, n-1$, we have*

$$\bar{V}_{1,i+1} = \bar{V}_{1,i} \cap g_{i,1}(E_{i+1}),$$

$$\bar{V}_{n,i-1} = \bar{V}_{n,i} \cap f_{i-1,n-1}(E_{i-1}),$$

and short exact sequences

$$0 \to \bar{V}_{1,i+1} \to \bar{V}_{1,i} \to \bar{Z}_{1,i} \to 0,$$

$$0 \to \bar{V}_{n,i-1} \to \bar{V}_{n,i} \to \bar{Z}_{n,i} \to 0.$$

Proof. The first two equalities follow immediately from the definitions, as do injectivity and surjectivity of the sequences. For exactness of the first sequence, we check the following identities:

$$\begin{aligned}
\bar{V}_{1,i+1} &= \bar{V}_{1,i} \cap g_{i,1}(E_{i+1}) \\
&= g_{i-1,1}(g_{i-1,1}^{-1}(V_1)) \cap g_{i-1,1}(g_i(E_{i+1})) \\
&= g_{i-1,1}(g_{i-1,1}^{-1}(V_1) \cap g_i(E_{i+1})) \\
&= g_{i-1,1}(g_{i-1,1}^{-1}(V_1) \cap \ker f_i) \\
&= \ker(\bar{V}_{1,i} \to \bar{Z}_{1,i}),
\end{aligned}$$

and exactness of the second sequence follows similarly. □

Notation 3.5. Given a point $\{V_i\}$ of LG, we denote by $V_{1,i} \subset V_1$ and $V_{n,i} \subset V_n$ the images $g_{i-1,1}(V_i)$ and $f_{i,n-1}(V_i)$ respectively, and by $Z_i \subset \bar{Z}_i$ the image of $V_{1,i} \oplus V_{n,i}$ in $\bar{Z}_i$.

Finally, for any vector space V, if we wish to prescribe a certain dimension for V, we will fix a non-negative integer which we will denote by d_V, and require that V satisfy $\dim(V) = d_V$; we will also abbreviate $\dim(V)$ by $\mathrm{d}(V)$.

Note that this is compatible with our earlier definitions of $V_{1,n}$, $V_{n,1}$.

Lemma 3.6. *With V_1, V_n given, let $V_{1,n}$ and $V_{n,1}$ be as determined by (V_1, V_n), and fix nested sequences of subspaces*

$$V_{1,n} \subset V_{1,n-1} \subset ... \subset V_{1,2} \subset V_1, \text{ and}$$

$$V_{n,1} \subset V_{n,2} \subset ... \subset V_{n,n-1} \subset V_n$$

with $V_{1,i} \subset \bar{V}_{1,i}$ and $V_{n,i} \subset \bar{V}_{n,i}$ for all i. Then points of the linked Grassmannian with the given V_1, V_n, and $V_{1,i}$ and $V_{n,i}$ correspond to collections of r-dimensional subspaces $V_i \subset \ker(V_{1,i} \oplus V_{n,i} \rightarrow \bar{Z}_i)$ with V_i mapping surjectively to $V_{1,i}$ and $V_{n,i}$ and containing $V_{1,i+1} \oplus V_{n,i-1}$ for $i = 2, \ldots, n-1$.

Proof. By Lemma 3.2, any $\{V_i\}$ corresponding to a point of the linked Grassmannian with given associated $V_{1,i}$ and $V_{n,i}$ may be considered as subspaces of $\ker(V_{1,i} \oplus V_{n,i} \rightarrow \bar{Z}_i)$, surjecting on $V_{1,i}$ and $V_{n,i}$. Conversely any V_i of this form may naturally be considered as a subspace of E_i. Thus, we need only check that the linking condition is equivalent to the containment of $V_{1,i+1} \oplus V_{n,i-1}$, which by Lemma 3.4 is contained in the kernel space in question.

We show that linking under the f_i is equivalent to V_i containing $(0) \oplus V_{n,i-1}$, with the argument for the g_i being the same, and the combination giving the desired statement. First, $f_1(V_1) \subset V_2$: indeed, the linked Grassmannian conditions and our definitions imply that $f_1(V_1) \xrightarrow{\sim} (0) \oplus V_{n,1} \subset E_1 \oplus E_n$, and $(0) \oplus V_{n,1} \subset V_1 \oplus V_n$ by hypothesis. Hence, $f_1(V_1)$ is contained in V_2 if and only if V_2 contains $(0) \oplus V_{n,1}$. Similarly, for $2 < i \leq n-1$, we have $f_{i-1}(V_{i-1}) \subset V_i$ if and only if V_i contains $(0) \oplus V_{n,i-1}$. Lastly, the containment $f_{n-1}(V_{n-1}) = V_{n,n-1} \subset V_n$ is automatic from our hypotheses. $\square$

Lemma 3.7. *Given a point $\{V_i\}$ of LG, each $V_{1,i}$ and $V_{n,i}$ maps surjectively to Z_i, so in particular $Z_i \subset \bar{Z}_{1,i} \cap \bar{Z}_{n,i}$.*

Proof. This is equivalent to the statement that the images of $V_{1,i}$ and $V_{n,i}$ agree inside $\bar{Z}_i$. But for any $v \in V_{1,i}$, by definition there is a vector $(v, v') \in V_{1,i} \oplus V_{n,i} \subset \bar{V}_{1,i} \oplus \bar{V}_{n,i}$ coming from V_i, hence mapping to 0 in Z_i. Thus $-v'$ has the same image as v in $\bar{Z}_i$, so the image of $V_{1,i}$ is contained in that of $V_{n,i}$. But the same argument works to show the opposite containment, giving the desired statement. $\square$

The final technical lemma is the following:

Lemma 3.8. *Fixing non-negative integer dimensions $d_{V_{1,i}}, d_{V_{n,i}}, d_{Z_i}$ for all i with $1 < i < n$ determines a locally closed subscheme of the fiber of the pr_{1n} map over (V_1, V_n).*

Proof. First, the conditions obtained by fixing the $d_{V_{1,i}}$ and $d_{V_{n,i}}$ determine a locally closed subscheme, since they are imposing a particular rank on the maps of the universal bundles $V_i \xrightarrow{g_{i-1,1}} V_1$ and $V_i \xrightarrow{f_{i,n-1}} V_n$, which are locally free. Next, within

the locally closed subscheme cut out by these conditions, we note that the $V_{1,i}$ and $V_{n,i}$ are also locally free (and the $\bar{Z}_i$, being determined by (V_1, V_n), are in fact free), so prescribing the ranks of the maps $V_{1,i} \to \bar{Z}_i$ (or equivalently, by Lemma 3.7, the ranks of the maps $V_{n,i} \to \bar{Z}_i$) determines locally closed conditions. $\square$

We thus obtain a stratification of the fiber of pr_{1n}, in the sense of having a collection of disjoint locally closed subschemes whose union is set-theoretically the entire fiber. Our main task will be to analyze this stratification further.

4. Dimensions of the fibers

We will initially analyze the pieces of our stratification to compute their dimensions. Altough we describe the fibers of pr_{1n} in terms of the dimensional invariants $\mathrm{d}(\bar{V}_{1,i}), \mathrm{d}(\bar{V}_{n,i}), \mathrm{d}(\bar{Z}_i)$, the formulas are quite complicated, and rather than work with them directly, we will be able to obtain an indirect bound by studying the dimension of pairs (V_1, V_n) having specified dimensional invariants. We can then use the fact that the dimensions of the fibers are determined entirely by the numerical invariants to obtain an indirect upper bound for them, Corollary 4.4 below.

Theorem 4.1. *The fibers of pr_{1n} have dimension determined by the dimensions $\mathrm{d}(V_{1,n})$, $\mathrm{d}(V_{n,1})$, $\mathrm{d}(\bar{V}_{1,n})$, $\mathrm{d}(\bar{V}_{n,1})$, and $\mathrm{d}(\bar{V}_{1,i})$, $\mathrm{d}(\bar{V}_{n,i})$, and $\mathrm{d}(\bar{Z}_i)$ for $1 < i < n$.*

Specifically, assuming (V_1, V_n) satisfies $f_{1,n-1}(V_1) \subset V_n, g_{n-1,1}V_n \subset V_1$, if we prescribe dimensions $d_{V_{1,i}}, d_{V_{n,i}}, d_{Z_i}$, the corresponding stratum of Lemma 3.8 is non-empty precisely when the following conditions are satisfied for all i with $2 \leq i \leq n-1$:

$$(4.1) \qquad\qquad d_{Z_i} \leq \mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n,i})$$

$$(4.2) \qquad\qquad \mathrm{d}(\bar{V}_{1,i+1}) \geq d_{V_{1,i}} - d_{Z_i} \geq d_{V_{1,i+1}}$$

$$(4.3) \qquad\qquad \mathrm{d}(\bar{V}_{n,i-1}) \geq d_{V_{n,i}} - d_{Z_i} \geq d_{V_{n,i-1}}$$

$$(4.4) \qquad\qquad d_{V_{1,i}} + d_{V_{n,i}} - d_{Z_i} \geq r$$

$$(4.5) \qquad\qquad r \geq d_{V_{1,i+1}} + d_{V_{n,i}}$$

$$(4.6) \qquad\qquad r \geq d_{V_{1,i}} + d_{V_{n,i-1}}$$

where by convention we write $d_{V_{n,1}} := \mathrm{d}(V_{n,1})$ and $d_{V_{1,n}} := \mathrm{d}(V_{1,1})$.

Each stratum is then smooth of dimension

$$(4.7) \quad \sum_{i=2}^{n-1} d_{Z_i}(\mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n,i}) - d_{Z_i})$$

$$+ (d_{V_{1,i}} - d_{V_{1,i+1}})(\mathrm{d}(\bar{V}_{1,i+1}) - d_{V_{1,i}} + d_{Z_i}) + (d_{V_{n,i}} - d_{V_{n,i-1}})(\mathrm{d}(\bar{V}_{n,i-1}) - d_{V_{n,i}} + d_{Z_i})$$

$$+ (r - d_{V_{1,i+1}} - d_{V_{n,i-1}})(d_{V_{1,i}} + d_{V_{n,i}} - d_{Z_i} - r).$$

Proof. We simply have to check that the dimensions (and emptyness or non-emptyness) of the pieces of our stratification are determined by the numbers in question. Note that by construction, for $2 \leq i \leq n-1$, $\bar{Z}_{1,i} + \bar{Z}_{n,i} = \bar{Z}_i$, so

$$\mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n,i}) = \mathrm{d}(\bar{Z}_{1,i}) + \mathrm{d}(\bar{Z}_{n,i}) - \mathrm{d}(\bar{Z}_i).$$

Furthermore, by Lemma 3.4 we have

$$\mathrm{d}(\bar{Z}_{1,i}) = \mathrm{d}(\bar{V}_{1,i}) - \mathrm{d}(\bar{V}_{1,i+1}), \text{ and } \mathrm{d}(\bar{Z}_{n,i}) = \mathrm{d}(\bar{V}_{n,i}) - \mathrm{d}(\bar{V}_{n,i-1}).$$

We fix non-negative integers $d_{V_{1,i}}, d_{V_{n,i}}, d_{Z_i}$, and consider the structure of the resulting stratum of Lemma 3.8, which we denote by X.

Denote by X' the functor of filtrations

$$g_{n-1,1}(V_n) = V_{1,n} \subset V_{1,n-1} \subset \ldots \subset V_{1,2} \subset V_1 \text{ and}$$

$$f_{1,n-1}(V_1) = V_{n,1} \subset V_{n,2} \subset \ldots \subset V_{n,n-1} \subset V_n$$

with each $V_{1,i} \subset \bar{V}_{1,i}$ and $V_{n,i} \subset \bar{V}_{n,i}$, and each of the prescribed dimension, with $V_{1,i}$ and $V_{n,i}$ having the same image, also of the prescribed dimension, in $\bar{Z}_i$. Also denote by X'' the functor of $(n-2)$-tuples of spaces $Z_i \subset \bar{Z}_{1,i} \cap \bar{Z}_{n,i}$ having the prescribed dimension. Then we have maps $X \to X' \to X''$, which we will analyze one by one.

X'' is simply a product of Grassmannians, hence a smooth scheme of dimension $\sum_{i=2}^{n-1} d_{Z_i}(\mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n,i}) - d_{Z_i})$, non-empty if and only if (4.1) is satisfied.

We next wish to show that X' is likewise representable by a scheme smooth over X'', of relative dimension

$$\sum_{i=n-1}^{2} (d_{V_{1,i}} - d_{V_{1,i+1}})(\mathrm{d}(\bar{V}_{1,i}) - d_{V_{1,i}} - \mathrm{d}(\bar{Z}_{1,i}) + d_{Z_i})$$

$$+ \sum_{i=2}^{n-1} (d_{V_{n,i}} - d_{V_{n,i-1}})(\mathrm{d}(\bar{V}_{n,i}) - d_{V_{n,i}} - \mathrm{d}(\bar{Z}_{n,i}) + d_{Z_i}),$$

and non-empty if and only if (4.2) and (4.3) are satisfied for i with $2 \le i \le n-1$. Having chosen the Z_i, the choices of $V_{1,i}$ are independent of the choices of $V_{n,i}$, and the situation is symmetric for both, so we treat only the former. We can describe X' as open inside a sequence of Grassmannian bundles over X'', with the first bundle corresponding to the choice of $V_{1,n-1}$, and each subsequent one corresponding to the choice of some $V_{1,i}$ given $V_{1,i+1}$.

Indeed, $V_{1,n-1}$ may be any space containing $V_{1,n}$, contained in the subspace of $\bar{V}_{1,n-1}$ mapping into Z_{n-1}, and surjecting onto Z_{n-1}. The last is an open condition, and we claim we have non-emptiness of dimension

$$(d_{V_{1,n-1}} - \mathrm{d}(V_{1,n}))(\mathrm{d}(\bar{V}_{1,n-1}) - d_{V_{1,n-1}} - (\mathrm{d}(\bar{Z}_{1,n-1}) - d_{Z_{n-1}}))$$
$$= (d_{V_{1,n-1}} - \mathrm{d}(V_{1,n}))(\mathrm{d}(\bar{V}_{1,n}) - d_{V_{1,n-1}} + d_{Z_{n-1}})$$

exactly when (4.2) is satisfied for $i = n-1$. Non-emptiness of the Grassmannian is equivalent to $\mathrm{d}(\bar{V}_{1,n}) + d_{Z_{n-1}} \ge d_{V_{1,n-1}} \ge d_{V_{1,n}}$. We note that by Lemma 3.4 we have that $V_{1,n}$ maps to 0 in $\bar{Z}_i$, so non-emptiness of the open condition of surjecting onto Z_{n-1} is then equivalent to $d_{V_{1,n-1}} - d_{V_{1,n}} \ge d_{Z_i}$, completing the proof of the claim. Similarly, for each $i < n-1$, we can choose $V_{i,1}$ to be any space containing $V_{1,i+1}$, contained in the subspace of $\bar{V}_{1,i}$ mapping into Z_i, and surjecting onto Z_i. This is (open inside) another Grassmannian bundle, of dimension $(d_{V_{1,i}} - d_{V_{1,i+1}})(\mathrm{d}(\bar{V}_{1,i}) - d_{V_{1,i}} - \mathrm{d}(\bar{Z}_{1,i}) + d_{Z_{1,i}})$ and non-empty exactly when (4.2) is satisfied for $i = n-2, \ldots, 2$. Arguing the same way for the $V_{n,i}$, we find that (4.3) is precisely the condition for non-emptiness, and we conclude the desired description of X'.

Finally, given the $V_{1,i}$ and $V_{n,i}$, by Lemma 3.6 we have that X is open inside a bundle of products of Grassmannians of dimension

$$\sum_{i=2}^{n-1}(r - d_{V_{1,i+1}} - d_{V_{n,i-1}})(d_{V_{1,i}} + d_{V_{n,i}} - d_{Z_i} - r)$$

over X'. Now, non-emptyness of the bundle is equivalent to (4.4) together with

$$r \geq d_{V_{1,i+1}} + d_{V_{n,i-1}},$$

and we claim that non-emptyness of the open condition that each V_i surject onto $V_{1,i}$ and $V_{n,i}$ is equivalent to (4.5) and (4.6). This claim is easily checked, recalling the conditions imposed on V_i by the containment of $V_{1,i+1} \oplus (0)$ and $(0) \oplus V_{n,i-1}$, by the observation that because each of $V_{1,i}$ and $V_{n,i}$ surject onto Z_i, we must have that $\ker(V_{1,i} \oplus V_{n,i} \to Z_i)$ surjects onto $V_{1,i}$ and $V_{n,i}$. Finally, either of (4.5) or (4.6) implies the above inequality, so we find that non-emptiness of X, given non-emptiness of X', is equivalent to (4.4), (4.5), and (4.6).

This shows that the dimension and non-emptyness of X are entirely determined by the various prescribed dimensions, and are otherwise independent of (V_1, V_n), completing the proof of the theorem. $\qquad\square$

Because of the complexity of the formula obtained from the preceding theorem, we will find it profitable to analyze the situation less directly, by considering spaces of spaces (V_1, V_n) with given dimensional invariants. We first need to introduce one more piece of notation:

Notation 4.2. We denote by $\tilde{Z}_i$ the cokernel of the map

$$E_i \hookrightarrow f_{i,n-1}(E_i) \oplus g_{i-1,1}(E_i).$$

Note that the injectivity of the map follows from the same argument as in Lemma 3.2. Note also that directly from the definition, one sees that each of $f_{i,n-1}(E_i)$ and $g_{i-1,1}(E_i)$ surjects onto $\tilde{Z}_i$.

Theorem 4.3. *The dimension of the space of pairs* $(V_1, V_n) \in G_1 \times G_n$ *having dimensions prescribed by fixed* $d_{\bar{V}_{1,i}}, d_{\bar{V}_{n,i}}$ *and* $d_{\bar{Z}_i}$ *for* $i = 2, \ldots, n-1$, *as well as* $d_{V_{1,n}}, d_{V_{n,1}}, d_{\bar{V}_{1,n}}$, *and* $d_{\bar{V}_{n,1}}$, *and satisfying* $f_{1,n-1}(V_1) \subset V_n$, $g_{n-1,1}(V_n) \subset V_1$, *is given by*

$$\sum_{i=1}^{n} d_{\bar{Z}_{1,i}}(d_{g_{i-1,1}(E_i)} - d_{\bar{V}_{1,i}}) + d_{\bar{Z}_{n,i}}(d_{f_{i,n-1}(E_i)} - d_{\bar{V}_{n,i}})$$

$$+ d_{V_{1,n}}(d_{\bar{V}_{1,n}} - d_{V_{1,n}}) + d_{V_{n,1}}(d_{\bar{V}_{n,1}} - d_{V_{n,1}}) - \sum_{i=2}^{n-1} d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}}(d_{\tilde{Z}_i} - d_{\bar{Z}_i}),$$

where we set

$$d_{\bar{Z}_{1,i}} = d_{\bar{V}_{1,i}} - d_{\bar{V}_{1,i+1}}, \text{ and } d_{\bar{Z}_{n,i}} = d_{\bar{V}_{n,i}} - d_{\bar{V}_{n,i-1}},$$

and

$$d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}} = d_{\bar{Z}_{1,i}} + d_{\bar{Z}_{n,i}} - d_{\bar{Z}_i}$$

for each $i = 2, \ldots, n-1$, *and use the conventions that*

$$d_{\bar{V}_{1,n+1}} = d_{\bar{V}_{n,0}} = 0,$$

$$d_{\bar{V}_{1,1}} = d_{\bar{V}_{n,n}} = r,$$

$$d_{g_{0,1}(E_1)} = d_{V_{n,1}} + d_{\ker f_1}, \ \ and$$
$$d_{f_{n,n-1}(E_1)} = d_{V_{1,n}} + d_{\ker g_{n-1}}.$$

This space is smooth, and non-empty if and only if the following conditions are satisfied for $i = 2, \ldots, n-1$:

(4.8) $$d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}} \leq d_{\bar{Z}_{1,i}} \leq d_{\tilde{Z}_i}$$

(4.9) $$d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}} \leq d_{\bar{Z}_{n,i}} \leq d_{\tilde{Z}_i}$$

(4.10) $$d_{\bar{Z}_{1,i}} + d_{\bar{Z}_{n,i}} \leq d_{\tilde{Z}_i} + d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}}$$

(4.11) $$d_{V_{n,1}} \leq d_{\bar{V}_{n,1}} \leq d_{f_{1,n-1}(E_1)}$$

(4.12) $$d_{V_{1,n}} \leq d_{\bar{V}_{1,n}} \leq d_{g_{n-1,1}(E_n)}$$

(4.13) $$d_{\bar{V}_{1,i+1}} \leq d_{\bar{V}_{1,i}} \leq d_{g_{i-1,1}(E_i)} + d_{\bar{Z}_{1,i}} - d_{\tilde{Z}_i}$$

(4.14) $$d_{\bar{V}_{n,i-1}} \leq d_{\bar{V}_{n,i}} \leq d_{f_{i,n-1}(E_i)} + d_{\bar{Z}_{n,i}} - d_{\tilde{Z}_i}$$

(4.15) $$d_{\bar{V}_{1,2}} + d_{V_{n,1}} \leq d_{\bar{V}_{1,1}} \leq d_{g_{0,1}(E_1)}$$

(4.16) $$d_{\bar{V}_{n,n-1}} + d_{V_{1,n}} \leq d_{\bar{V}_{n,n}} \leq d_{f_{n,n-1}(E_n)}$$

Proof. The proof proceeds similarly to that of Theorem 4.1: we build up V_1 by one $\bar{V}_{1,i}$ at a time, and similarly for V_n. However, we will have to begin by choosing the $\bar{Z}_{1,i} \cap \bar{Z}_{n,i}$, followed by the $\bar{Z}_{1,i}$ and $\bar{Z}_{n,i}$. If we denote by X the space of pairs (V_1, V_n) of the appropriate form, we will want to consider the functors X' of $(2n-4)$-tuples of $\bar{Z}_{1,i}$ and $\bar{Z}_{n,i}$ inside $\tilde{Z}_i$ such that $d(\bar{Z}_{1,i} \cap \bar{Z}_{n,i}) = d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}}$, and X'' of $(n-2)$-tuples of $\bar{Z}_{1,i} \cap \bar{Z}_{n,i}$ inside $\tilde{Z}_i$ of dimension $d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}}$. As before, we have natural maps $X \to X' \to X''$ which we analyze one at a time.

Now, X'' is clearly represented by a product of Grassmannians of total dimension

$$\sum_{i=2}^{n-1} d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}} (d_{\tilde{Z}_i} - d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}}),$$

non-empty if and only if $d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}} \leq d_{\tilde{Z}_i}$, which is implied by (4.8). X' is then open inside a bundle in products of Grassmannians over X'', of dimension

$$\sum_{i=2}^{n-1} (d_{\bar{Z}_{1,i}} - d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}})(d_{\tilde{Z}_i} - d_{\bar{Z}_{1,i}}) + (d_{\bar{Z}_{n,i}} - d_{\bar{Z}_{1,i} \cap \bar{Z}_{n,i}})(d_{\tilde{Z}_i} - d_{\bar{Z}_{n,i}});$$

the bundle is non-empty if and only if (4.8) and (4.9) are satisfied, while the open condition that the intersection of $\bar{Z}_{1,i}$ and $\bar{Z}_{n,i}$ is no larger than the space chosen for $\bar{Z}_{1,i} \cap \bar{Z}_{n,i}$ is non-empty if and only if (4.10) is satisfied.

Finally, we show that X is made up of a tower of open subschemes of Grassmannian bundles over X', by building up V_1 and V_n starting from $V_{1,n}$ and $V_{n,1}$ and continuing up through the $\bar{V}_{1,i}$ and $\bar{V}_{n,i}$. The conditions on the $\bar{V}_{1,i}$ and $\bar{V}_{n,i}$ as we build them up will be simply that for each $i = 2, \ldots, n-1$, we have $\bar{V}_{1,i} \cap g_{i,1}(E_{i+1}) = \bar{V}_{1,i+1}$, and $\bar{V}_{n,i} \cap f_{i-1,n-1}(E_{i-1}) = \bar{V}_{n,i-1}$. Of course, we will also require that $V_{1,n} = g_{n-1,1}(V_n)$ and $V_{n,1} = f_{1,n-1}(V_1)$. These conditions will ensure that the $\bar{V}_{1,i}$ and $\bar{V}_{n,i}$ in fact come from (V_1, V_n) as prescribed by Notation 3.1.

The spaces $\bar{V}_{1,i}$ are almost independent from the $\bar{V}_{n,i}$, except for the final requirement on $V_{1,n}$ and $V_{n,1}$; however, since all $\bar{V}_{1,i}$ for $i > 1$ are by definition in the image of g_1, they map to 0 under f_1 and hence $f_{1,n-1}$, and similarly for the $\bar{V}_{n,i}$,

so this dependence will only appear when we choose $V_1 = \bar{V}_{1,1}$ and $V_n = \bar{V}_{n,n}$, after all the previous $\bar{V}_{1,i}$ and $\bar{V}_{n,i}$ have been chosen.

First, choosing $V_{1,n} \subset g_{n-1,1}(E_n)$ and $V_{n,1} \subset f_{1,n-1}(E_1)$ is clearly a product of Grassmannians of dimension

$$d_{V_{1,n}}(d_{g_{n-1,1}(E_n)} - d_{V_{1,n}}) + d_{V_{n,1}}(d_{f_{1,n-1}(E_1)} - d_{V_{n,1}}),$$

non-empty as long as (4.11) and (4.12) are satisfied. Next, choosing $\bar{V}_{1,n} \subset g_{n-1,1}(E_n)$ containing $V_{1,n}$, and $\bar{V}_{n,1} \subset f_{1,n-1}(E_1)$ containing $V_{n,1}$ is again a product of Grassmannians, of dimension

$$(d_{\bar{V}_{1,n}} - d_{V_{1,n}})(d_{g_{n-1,1}(E_n)} - d_{\bar{V}_{1,n}}) + (d_{\bar{V}_{n,1}} - d_{V_{n,1}})(d_{f_{1,n-1}(E_1)} - d_{\bar{V}_{n,1}}),$$

non-empty as long as (4.11) and (4.12) are satisfied; moreover, we see that to have non-emptiness in both cases, we must have (4.11) and (4.12).

For $i = n-1, \ldots, 2$, we allow $\bar{V}_{1,i}$ to be an arbitrary subspace of the preimage of our chosen $\bar{Z}_{1,i}$ inside $g_{i-1,1}(E_i)$, which must contain $\bar{V}_{1,i+1}$, map surjectively onto $\bar{Z}_{1,i}$, and must intersect with $g_{i,1}(E_{i+1})$ in precisely $\bar{V}_{1,i+1}$. Because each $g_{i-1,1}(E_i)$ surjects onto $\tilde{Z}_i$, this will be open inside a Grassmannian of dimension

$$(d_{\bar{V}_{1,i}} - d_{\bar{V}_{1,i+1}})(d_{g_{i-1,1}(E_i)} - (d_{\tilde{Z}_i} - d_{\bar{Z}_{1,i}}) - d_{\bar{V}_{1,i}}).$$

This Grassmannian is non-empty if and only if (4.13) is satisfied, so it remains to analyze the open conditions, which we claim are always non-empty.

Noting that $g_{i,1}(E_{i+1}) = \ker(g_{i-1,1}(E_i) \to \tilde{Z}_i)$, one checks that the condition of surjecting onto $\bar{Z}_{1,i}$ is equivalent to the condition that $\bar{V}_{1,i}$ intersect $g_{i,1}(E_{i+1})$ exactly in $\bar{V}_{1,i+1}$, and the non-emptiness of both is equivalent to the inequality $d_{\bar{V}_{1,i}} - d_{\bar{V}_{1,i+1}} \geq d_{\bar{Z}_{1,i}}$, which we have imposed as a condition of the theorem (and which is automatically satisfied if one starts with a pair (V_1, V_n) by Lemma 3.4).

The situation for the $\bar{V}_{n,i}$ is the same, contributing dimensions of

$$(d_{\bar{V}_{n,i}} - d_{\bar{V}_{n,i-1}})(d_{f_{i,n-1}(E_i)} - (d_{\tilde{Z}_i} - d_{\bar{Z}_{n,i}}) - d_{\bar{V}_{n,i}})$$

at each step, and non-empty if and only if (4.14) is satisfied.

Finally, we need to choose V_1 containing $\bar{V}_{1,2}$ and mapping surjectively onto $V_{n,1}$ under $f_{1,n-1}$. This is open inside a Grassmannian of dimension $(r - d_{\bar{V}_{1,2}})(d_{V_{n,1}} + d_{\ker f_1} - r)$, with the Grassmannian non-empty if (following our notational conventions) $d_{\bar{V}_{1,2}} \leq d_{\bar{V}_{1,1}} \leq d_{g_{0,1}(E_1)}$, and the open condition non-empty if $d_{\bar{V}_{1,2}} + d_{V_{n,1}} \leq d_{\bar{V}_{1,1}}$; together, they are non-empty if and only if (4.15) is satisfied. Similarly, choosing V_n is open inside a Grassmannian of dimension $(r - d_{\bar{V}_{n,n-1}})(d_{V_{1,n}} + d_{\ker g_{n-1}} - r)$, non-empty when (4.16) is satisfied. This describes X completely, and adding up the dimension formulas and cancelling terms gives the asserted formula and completes the proof of the theorem. $\square$

Corollary 4.4. *The dimension of a fiber of pr_{1n} over a point (V_1, V_n) is at most equal to*

$$r(d-r) - \mathrm{d}(V_{1,n})(\mathrm{d}(\bar{V}_{1,n}) - \mathrm{d}(V_{1,n})) - \mathrm{d}(V_{n,1})(\mathrm{d}(\bar{V}_{n,1}) - \mathrm{d}(V_{n,1}))$$

$$+ \sum_{i=2}^{n-1} \mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n,i})(\mathrm{d}(\tilde{Z}_i) - \mathrm{d}(\bar{Z}_i))$$

$$- \sum_{i=1}^{n} \left(\mathrm{d}(\bar{Z}_{1,i})(\mathrm{d}(g_{i-1,1}(E_i)) - \mathrm{d}(\bar{V}_{1,i})) + \mathrm{d}(\bar{Z}_{n,i})(\mathrm{d}(f_{i,n-1}(E_i)) - \mathrm{d}(\bar{V}_{n,i})) \right).$$

Proof. This follows immediately from the two theorems together with Theorem 2.2, which implies that the dimension of LG is $r(d-r)$. □

5. Applications to limit linear series

In this section, we describe the promised applications to the theory of limit linear series on curves. First, we recall the basic theorems on spaces of limit series, and how the linked Grassmannian is used to construct them (see [7, §5] for the general case, and additional details).

We state the general theorems first, and then recall in more detail the situation for a reducible curve over a field.

Situation 5.1. Let X/B, together with smooth sections $P_1, \ldots, P_n$, be a **smoothing family**: B should be regular, and X should be flat and proper over B, with fibers which are at worst nodal curves; for the full technical details, see [7, Def. 3.1]. We further assume that X/B has at most one node.

In [7], for integers r, d and ramification sequences $\alpha^1, \ldots, \alpha^n$, we describe a functor $\mathscr{G}^r_d$ of relative limit series of degree d and dimension r on X/B, ramified to order at least α^i at each P_i. This functor is compatible with base change, and agrees with usual linear series on the smooth fibers of X/B. We have the following basic theorem:

Theorem 5.2. [7, Thm. 5.3] *In the above situation, the functor $\mathscr{G}^r_d(X)$ is represented by a scheme $G^r_d(X)$, compatible with base change to any other smoothing family. This scheme is projective over B, and if it is non-empty, the local ring at any point $x \in G^r_d(X)$ closed in its fiber over $b \in B$ has dimension at least $\dim \mathscr{O}_{B,b} + \rho$, where $\rho = (r+1)(d-r) - rg - \sum_{i,j} \alpha^i_j$.*

We now specialize to the case that that X is over $\operatorname{Spec} k$, with two smooth components Y and Z glued at a single node Δ'. We fix smooth points $P_1, \ldots, P_n$ of X, as well as integers r, d, and ramification sequences $\alpha^1, \ldots, \alpha^n$. In this situation, our functor roughly parametrizes line bundles $\mathscr{L}$ on X of degree d on Y and degree 0 on Z, together with vector spaces $V_i \subset H^0(X, \mathscr{L}^i)$ of dimension r, where:

  (i) We define $\mathscr{L}^i$ to be the line bundle obtained by gluing together $(\mathscr{L}|_Y)(-i\Delta')$ and $(\mathscr{L}|_Z)(i\Delta')$;
  (ii) Each V_i maps into V_{i+1} under the natural map $\mathscr{L}^i \to \mathscr{L}^{i+1}$ induced by the natural inclusion on Z and the zero map on Y, and each V_{i+1} maps similarly into V_i.

We then construct the space $G^r_d(X)$ as a closed subscheme of a linked Grassmannian as follows.

First, choose a divisor D of very large degree, supported non-trivially on both Y and Z, and disjoint from the P_i and Δ'. Although the construction depends on the choice of D, the resulting scheme represents the functor $\mathscr{G}^r_d(X)$, which is described independently of D. For $i = 0, \ldots, d$ let $P^i := \operatorname{Pic}^{d-i,i}(X) \cong \operatorname{Pic}^{d-i}(Y) \times \operatorname{Pic}^i(Z)$ denote the Picard scheme of line bundles on X restricting to degree $d - i$ on Y and degree i on Z. Let $\mathscr{L}^i$ be the universal line bundle on $P^i \times X$, and fix isomorphisms between the P^i so that we can consider the $\mathscr{L}^i$ as line bundles on a single scheme $P \times X$. We then have maps $\mathscr{L}^i \to \mathscr{L}^{i+1}$ and $\mathscr{L}^{i+1} \to \mathscr{L}^i$ as described above.

For $i = 0, \ldots, d$, we thus obtain maps $f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$ and $g_i : \mathscr{E}_{i+1} \to \mathscr{E}_i$, where $\mathscr{E}_i := p_{1*}(\mathscr{L}^{i-1}(D))$, with $p_1 : P \times X \to P$ the projection map. We then get

a linked Grassmannian LG of length $n' = d + 1$ over P, looking at sub-bundles of the $\mathscr{E}_i$ of rank $r' = r + 1$. We denote by d' the rank of the $\mathscr{E}_i$, so that LG has relative dimension $r'(d' - r')$ over P. Writing $D = D^Y + D^Z$ where D^Y and D^Z are supported on Y and Z respectively, we then obtain our $G_d^r(X)$ space as a closed subscheme of LG by requiring that the space of sections in $\mathscr{E}_1 = p_{1*}(\mathscr{L}^0(D))$ vanishes along D^Y, and the space in $\mathscr{E}_{n'} = p_{1*}(\mathscr{L}^d(D))$ vanishes along D^Z, which together imply that all sections will vanish along D, and hence come from the original $\mathscr{L}^i$. Finally, the ramification conditions at the P_i are likewise imposed on $\mathscr{E}_1$ or $\mathscr{E}_{n'}$ depending on whether P_i lies on Y or Z.

Given a point of $G_d^r(X)$ described by $\mathscr{L}$ and $V_0, \ldots, V_d$, we obtain a pair, which we denote by (V^Y, V^Z), of linear series of degree d and dimension r on Y and Z, by taking $V^Y = V_0|_Y$, $V^Z = V_d|_Z$ (as spaces of sections of the line bundles $\mathscr{L}^Y := \mathscr{L}^0|_Y, \mathscr{L}^Z := \mathscr{L}^d|_Z$, which we omit from the notation). We thus have as natural map $FR : G_d^r(X) \to G_d^r(Y) \times G_d^r(Z)$. The limit linear series defined by Eisenbud and Harris can be considered to be the closed subscheme $G_d^{r,\mathrm{EH}}(X) \subset G_d^r(Y) \times G_d^r(Z)$ cut out by the condition that, if a_j^Y and a_j^Z denote the vanishing sequences at the node of V^Y and V^Z respectively, for $j = 0, \ldots, r$ we have

(5.1) $$ a_j^Y + a_{r-j}^Z \geq d. $$

An Eisenbud-Harris limit series is refined when these inequalities are all equalities. In [7] the following theorem is proved:

Theorem 5.3. [7, Prop. 6.6, Cor. 6.8] *The natural map $FR : G_d^r(X) \to G_d^r(Y) \times G_d^r(Z)$ has set-theoretic image consisting precisely of the space of Eisenbud-Harris limit series. This map is an isomorphism when restricted to the open subset of $G_d^r(X)$ mapping to refined Eisenbud-Harris limit series.*

In particular, we see that $G_d^r(X)$ is obtained from LG simply by imposing conditions on the first and last projection maps, so our computations of fiber dimension for LG will also apply to $G_d^r(X)$, with fibers of pr_{1n} for LG corresponding precisely to fibers of FR for $G_d^r(X)$. Since we are working with such fibers, although LG will be over the non-trivial base P, to study any given fiber of pr_{1n} we can restrict to the corresponding point on the base, which is equivalent to fixing our choice of the line bundle $\mathscr{L}$. We now describe the relationship between the various numerical invariants in the two situations.

Lemma 5.4. *In the situation described above, and using the notational conventions of Theorem 4.3, for a given Eisenbud-Harris limit series (V^Y, V^Z), with vanishing sequences a^Y and a^Z at Δ', we have the following dimension formulas for the corresponding pair $(V_1, V_{n'}) \in G_1 \times G_{n'}$:*

$$ \mathrm{d}(\bar{V}_{1,i}) = \#\{j : a_j^Y \geq i - 1\} \text{ for } i = 1, \ldots, n'; $$

$$ \mathrm{d}(\bar{V}_{n',i}) = \#\{j : a_j^Z \geq n' - i\} \text{ for } i = 1, \ldots, n'; $$

$$ \mathrm{d}(g_{i-1,1}(E_i)) = d + \deg D^Y + 1 - g_Y - i + 1 \text{ for } i = 2, \ldots, n'; $$

$$ \mathrm{d}(f_{i,n'-1}(E_i)) = d + \deg D^Z + 1 - g_Z - (n' - i) \text{ for } i = 1, \ldots, n' - 1; $$

$$ \mathrm{d}(\tilde{Z}_i) = 1 \text{ for } i = 2, \ldots, n - 1. $$

In addition, we have the following statements at the boundaries:

$$ \mathrm{d}(V_{1,n'}) = \#\{j : a_j^Z \leq 0\}; $$

$$\mathrm{d}(V_{n',1}) = \#\{j : a_j^Y \leq 0\};$$

$$\mathrm{d}(g_{0,1}(E_1)) = d + \deg D^Y + 1 - g_Y - 1 + \mathrm{d}(V_{n',1});$$

$$\mathrm{d}(f_{n',n'-1}(E_{n'})) = d + \deg D^Z + 1 - g_Z - 1 + \mathrm{d}(V_{1,n'}).$$

Proof. We first note that (V^Y, V^Z) completely determine V_0 and V_d on X: $\mathscr{L}^0|_Z$ has degree 0, so any section in V^Y vanishing at Δ' extends over Z only by zero, while if a section is non-vanishing at Δ', we have $a_0^Y = 0$, so $a_r^Z = d$ by (5.1), and we must have $\mathscr{L}^Z \cong \mathscr{O}_Z(d\Delta')$, so $\mathscr{L}^0|_Z$ is the trivial bundle, and we can extend uniquely by a constant. The same argument shows that V^Z uniquely determines V_d.

Now, by definition, for all i we have that $\bar{V}_{1,i}$ is the space of sections of V^0 in the image of $H^0(X, \mathscr{L}^{i-1}(D))$, which is exactly the space of sections of $\mathscr{L}$ vanishing to order at least $i-1$ at Δ', giving the asserted dimension. We defined $V_{1,n'}$ to be the space of sections of V^0 in the image of V^d; we note that this is zero-dimensional if V^d has no sections non-vanishing on Y, and one-dimensional otherwise, giving the asserted formula for $\mathrm{d}(V_{1,n'})$. The formulas for $\mathrm{d}(\bar{V}_{n',i})$ and $\mathrm{d}(V_{n',1})$ are obtained similarly.

Next, $g_{i-1,1}(E_i)$ for $i \geq 2$ are those sections of E_1 in the image of E_i; such sections necessarily vanish on Z, and are in fact in correspondence with sections of $\mathscr{L}^0(D)|_Y$ vanishing to order at least $i-1$ at Δ'. Because D was chosen sufficiently large, $h^0(Y, \mathscr{L}^0(D)|_Y) = \deg \mathscr{L}^0(D)|_Y + 1 - g_Y = d + \deg D^Y + 1 - g_Y$, and the number of sections vanishing to order at least $i-1$ at Δ' is $d + \deg D^Y + 1 - g_Y - i + 1$, as asserted. For $i = 1$, by our notational conventions we need to consider $\mathrm{d}(\bar{V}_{n,1}) + \mathrm{d}(\ker f_1)$. We have $\ker f_1 = g_1(E_1)$, so we know this has dimension $d + \deg D^Y + 1 - g_Y - 1$, and we obtain the asserted formula. The same argument works to compute $\mathrm{d}(f_{i,n'}(E_i))$.

Finally, we have $\mathrm{d}(\tilde{Z}_i) \leq 1$ for all i because given a line bundle $\mathscr{L}$ on X, the only obstruction to gluing sections of $\mathscr{L}|_Y$ and $\mathscr{L}|_Z$ to obtain a section of $\mathscr{L}$ is whether the sections agree in $\mathscr{L}|_{\Delta'} \cong k$. On the other hand, we have $\mathrm{d}(\tilde{Z}_i) \geq 1$ because for $\mathscr{L} = \mathscr{L}^i(D)$, both restriction maps $\mathscr{L}|_Y \to \mathscr{L}_{\Delta'}$ and $\mathscr{L}|_Z \to \mathscr{L}_{\Delta'}$ are surjective because D was chosen to be sufficiently ample. $\square$

Corollary 5.5. *The dimension of the space of crude limit series corresponding to a given Eisenbud-Harris limit series (V^Y, V^Z) is bounded above by*

$$\sum_{i=0}^{r} (a_i^Y + a_{r-i}^Z - d).$$

Proof. This is a direct application of Corollary 4.4, together with the preceding lemma. We begin by recasting the formula

$$\sum_{i=1}^{n} \left(\mathrm{d}(\bar{Z}_{1,i})(\mathrm{d}(g_{i-1,1}(E_i)) - \mathrm{d}(\bar{V}_{1,i})) + \mathrm{d}(\bar{Z}_{n',i})(\mathrm{d}(f_{i,n'-1}(E_i)) - \mathrm{d}(\bar{V}_{n',i})) \right)$$

$$+ \mathrm{d}(V_{1,n'})(\mathrm{d}(\bar{V}_{1,n'}) - \mathrm{d}(V_{1,n'})) + \mathrm{d}(V_{n',1})(\mathrm{d}(\bar{V}_{n',1}) - \mathrm{d}(V_{n',1}))$$

$$- \sum_{i=2}^{n'-1} \mathrm{d}(\bar{Z}_{1,i} \cap \bar{Z}_{n',i})(\mathrm{d}(\tilde{Z}_i) - \mathrm{d}(\bar{Z}_i)),$$

in terms of the numerical invariants of (V^Y, V^Z).

We first see from the lemma that $d(\tilde{Z}_i) = 1$, and since $d(\bar{Z}_{1,i} \cap \bar{Z}_{n',i}) \le d(\bar{Z}_i)$, the term $\sum_{i=2}^{n'-1} d(\bar{Z}_{1,i} \cap \bar{Z}_{n',i})(d(\tilde{Z}_i) - d(\bar{Z}_i))$ always vanishes. Similarly, we have $d(\bar{V}_{1,n'}) = \#\{j : a_j^Y \ge d\}$, so is at most 1, and likewise $d(\bar{V}_{n',1}) \le 1$, so we have that the terms $d(V_{1,n'})(d(\bar{V}_{1,n'}) - d(V_{1,n'})) + d(V_{n',1})(d(\bar{V}_{n',1}) - d(V_{n',1}))$ always vanish. Thus, it is enough to consider the first sum.

We now claim that we have:

$$\sum_{i=1}^{n} d(\bar{Z}_{1,i})(d(g_{i-1,1}(E_i)) - d(\bar{V}_{1,i})) + d(\bar{Z}_{n,i})(d(f_{i,n-1}(E_i)) - d(\bar{V}_{n,i}))$$
$$= r'(d' - r') - \sum_{j=0}^{r}(a_j^Y + a_{r-j}^Z - d).$$

Indeed, we see that

$$d(\bar{Z}_{1,i}) = d(\bar{V}_{1,i}) - d(\bar{V}_{1,i+1}) = \begin{cases} 1 : \exists j \text{ such that } a_j^Y = i - 1 \\ 0 : \text{otherwise} \end{cases},$$

so if we split the sum in two, the first sum may be rewritten as

$$\delta_Y + \sum_{j=0}^{r}(d + \deg D^Y + 1 - g_Y - a_j^Y - (r + 1 - j)),$$

and similarly the second sum is

$$\delta_Z + \sum_{j=0}^{r}(d + \deg D^Z + 1 - g_Z - a_j^Z - (r + 1 - j)),$$

where the δ_Y and δ_Z account for the possible discrepency between $d(g_{0,1}(E_1))$ and $d + \deg D^Y + 1 - g_Y$ and between $d(f_{n',n'-1}(E_{n'}))$ and $d + \deg D^Z + 1 - g_Z$ respectively. However, we claim that in fact $\delta_Y = \delta_Z = 0$. Indeed, applying the boundary cases of the lemma we have that $d(g_{0,1}(E_1)) = d + \deg D^Y + 1 - g_Y$ unless $d(V_{n,1}) = 0$, in which case $a_0^Y > 0$. But the term $(d(\bar{V}_{1,1}) - d(\bar{V}_{1,2}))(d(g_{0,1}(E_1)) - d(\bar{V}_{1,1}))$ only contributes to the sum if some $a_j^Y = 0$, or equivalently, if $a_0^Y = 0$. Thus, this term only appears in the sum if $d(g_{0,1}(E_1)) = d + \deg D^Y + 1 - g_Y$, and we have $\delta^Y = 0$. The same argument shows that $\delta^Z = 0$.

Next, noting that we can replace $(r + 1 - j)$ in the first sum by $(1 + j)$, combining them we get

$$(r + 1)(d + \deg D^Y + \deg D^Z - g_Y - g_Z) - \sum_{j=0}^{r}(a_j^Y + a_{r-j}^Z - d + 2j)$$
$$= r'(d + \deg D - g) - r(r + 1) - \sum_{j=0}^{r}(a_j^Y + a_{r-j}^Z - d)$$
$$= r'(d + \deg D + 1 - g - (r + 1)) - \sum_{j=0}^{r}(a_j^Y + a_{r-j}^Z - d)$$
$$= r'(d' - r') - \sum_{j=0}^{r}(a_j^Y + a_{r-j}^Z - d),$$

as desired.

Finally, putting these statements together and applying Corollary 4.4, we obtain the desired bound. $\square$

This allows us to conclude that if the components of a reducible curve have the expected dimensions of (ramified) linear series, then the curve as a whole has the expected dimension of limit linear series:

Corollary 5.6. *Let X be a proper curve of genus g over $\operatorname{Spec} k$, consisting of the union of two smooth components Y and Z at a point Δ'. Also fix distinct marked points $P_1, \ldots, P_n$ in the smooth locus of X. Given integers d, r, and ramification sequences $\alpha^i := \alpha_0^i, \ldots, \alpha_r^i$, suppose that the following condition holds:*

For any $\alpha^{\Delta'} := \alpha_0^{\Delta'}, \ldots, \alpha_r^{\Delta'}$, the space of linear series on Y of degree d and dimension r with ramification sequence at least α^i at any P_i lying on Y, and at least $\alpha^{\Delta'}$ at Δ', has the expected dimension $\rho^Y := (r+1)(d-r) - rg_Y - \sum_{P_i \in Y, j} \alpha_j^i - \sum_j \alpha_j^{\Delta'}$ if it is non-empty, and similarly for Z.

Then the space $G_d^r(X)$ of limit linear series on X is pure of dimension $\rho^X := (r+1)(d-r) - rg - \sum_{i,j} \alpha_j^i$, and non-empty if and only if there exist ramification sequences α^Y and α^Z with $\alpha_j^Y + \alpha_{r-j}^Z = d - r$ for each j such that the corresponding spaces $G_d^r(Y)$ and $G_d^r(Z)$ are both non-empty.

Proof. By Theorem 5.3, we know that the set-theoretic image of $G_d^r(X)$ in the space $G_d^r(Y) \times G_d^r(Z)$ is precisely the closed subscheme of pairs whose vanishing sequences $a_j^{\Delta', Y}$ and $a_j^{\Delta', Z}$ at Δ' satisfy the inequalities

$$a_j^{\Delta', Y} + a_{r-j}^{\Delta', Z} \geq d,$$

or equivalently, whose ramification sequences satisfy

$$\alpha_j^{\Delta', Y} + \alpha_{r-j}^{\Delta', Z} \geq d - r.$$

If we fix ramification sequences meeting this condition, we find that the dimension of the locus in $G_d^r(Y) \times G_d^r(Z)$ is at most

$$\rho^Y + \rho^Z =$$

$$(r+1)(d-r) - rg_Y - \sum_{P_i \in Y, j} \alpha_j^i - \sum_j \alpha_j^{\Delta', Y}$$

$$+ (r+1)(d-r) - rg_Z - \sum_{P_i \in Z, j} \alpha_j^i - \sum_j \alpha_j^{\Delta', Z}$$

$$= 2(r+1)(d-r) - r(g_Y + g_Z) - \sum_{i,j} \alpha_j^i - \sum_j (\alpha_j^{\Delta', Y} + \alpha_{r-j}^{\Delta', Z})$$

$$= \rho^X - \sum_j (\alpha_j^{\Delta', Y} + \alpha_{r-j}^{\Delta', Z} - (d-r)) \leq \rho^X$$

because of the above inequality and the identity $g_Y + g_Z = g$. We see further that to show that $\dim G_d^r(X) \leq \rho^X$, it suffices to see that the dimension of the fibers of the map $G_d^r(X) \to G_d^r(Y) \times G_d^r(Z)$ are at most $\sum_j (\alpha_j^{\Delta', Y} + \alpha_{r-j}^{\Delta', Z} - (d-r)) = \sum_j (a_j^Y + a_{r-j}^Z - d)$, which is precisely Corollary 5.5. But we have $\dim G_d^r(X) \geq \rho^X$ by Theorem 5.2, so we obtain equality. Finally, for the non-emptyness assertion, we note that even if the only Eisenbud-Harris limit series on X are crude, we can

impose weaker ramification conditions at Δ' to satisfy the desired equality, and we will still have that the corresponding spaces $G_d^r(Y)$ and $G_d^r(Z)$ are non-empty. $\square$

The base cases for our induction will be the following easy and well-known statements:

Lemma 5.7. *Let C be a smooth proper curve of genus g over $\operatorname{Spec} k$, with distinct marked points $P_1, \ldots, P_n$. Then for any r, d, and ramification sequences $\alpha^i := \alpha_0^i, \ldots, \alpha_r^i$, the space of linear series of degree d and dimension r on C having ramification sequences at least α^i at P_i for all i has dimension exactly $\rho := (r + 1)(d - r) - rg - \sum_{i,j} \alpha_j^i$ (but is not necessarily non-empty) in either of the following cases:*

    (i) *if $g = 0$ and $\operatorname{char} k = 0$;*
    (ii) *if $g = 1$ and $n = 1$.*

Assuming $\rho \geq 0$, case (ii) is non-empty if and only if the single imposed vanishing sequence does not have $a_r = d$, $a_{r-1} = d - 1$. Non-emptiness of case (i) is determined by Schubert calculus, but in particular is non-empty whenever every imposed ramification sequence is of the form $0, 1, \ldots, 1$.

Note that no generality hypotheses are required here. (i) fails in positive characteristic even for $r = 1$, although it may be proved in certain non-trivial cases.

Proof. The dimension statement for (i) is proved by an inductive argument and the Plücker formula; see [1, Thm. 2.3]. Non-emptiness is simply a question of Schubert calculus, and we recall the argument: the ramification conditions of the form $0, 1, \ldots, 1$ correspond to an intersection of Schubert classes $\sigma_{0,1,\ldots,1}$ in the Grassmannian $\mathbb{G}(r, d)$. If we pass to the dual, we obtain a collection of special Schubert classes $\sigma_{0,\ldots,0,r}$ in $\mathbb{G}(d - r - 1, d)$, and can easily check non-emptiness as long as the expected dimension is non-negative by inductively applying Pieri's formula [3, p. 271].

(ii) may be seen as follows: let the prescribed vanishing sequence at P_1 be $a_0, \ldots, a_r$. The expected dimension is $\rho = (r + 1)(d - r) - r - \sum_{i=0}^r (a_i - i)$.

We first consider the case that $a_r = d$. Then the only possibility is a linear series contained in $H^0(C, \mathscr{O}(dP_1))$. This space is d-dimensional, with sections vanishing to every order at P_1 except $d - 1$. Thus the space of linear series is contained in a Grassmannian $\mathbb{G}(r, d - 1)$, of dimension $(r + 1)(d - r - 1)$, and the ramification condition cuts out a Schubert cycle of codimension $\sum_{i=0}^{r-1}(a_i - i) + (a_r - r - 1)$ as long as $a_{r-1} < d - 1$; if $a_{r-1} = d - 1$, the space is necessarily empty. We thus obtain the desired statement in this case.

If $a_r < d$, the line bundle could be any line bundle of degree d; these are all of the form $\mathscr{L} = \mathscr{O}((d + 1)P_1 - Q)$ as Q varies over the points of C. We have $h^0(C, \mathscr{L}) = d$, and there are sections vanishing to all orders less than $d - 1$ at P_1. When $Q = P_1$, the last order of vanishing is d, while for $Q \neq P_1$, the last is $d - 1$. The ramification condition is imposed inside a $\mathbb{G}(r, d)$-bundle over $\operatorname{Pic}(C) \cong C$, of dimension $1 + (r + 1)(d - r - 1)$; since we only need an upper bound on the dimension, we may work fiber by fiber. We claim that for each $\mathscr{L}$, the ramification condition imposes a Schubert cycle of codimension $\sum_{i=0}^r (a_i - i)$, giving the correct dimension for the total space. This is clear if $a_r < d - 1$, or if $a_r = d - 1$ and $Q \neq P_1$. In the last case, we note that the condition imposed by $a_r = d - 1$ is the

same as that imposed by $a_r = d$, so we still get a Schubert cycle of the asserted dimension.

Thus, we see that the space always has the asserted dimension, and is non-empty as long as we do not have $a_r = d, a_{r-1} = d - 1$. □

We are now ready for:

Proof of Theorem 1.1. As promised, everything except the connectedness statement will follow immediately from the limit linear series machinery; we will conclude connectedness in the reducible case from the theorem of Fulton and Lazarsfeld in the irreducible case. We first note that properness and the lower bound on the dimension follow directly from Theorem 5.2, and require no generality or characteristic hypotheses. We will prove the upper bound on dimension, and non-emptiness by induction on g.

In fact, we induct on a slightly stronger statement: we will show non-emptiness for $\rho \geq 0$ also in the case that we have imposed ramification sequences $0, 1, \ldots, 1$ at general points. The base case is $g = 0$; this is case (i) of Lemma 5.7. Suppose we now know the statement for curves of genus less than g, and we want to conclude it for curves of genus g. We first note that the reducible case for any curve with both components of genus strictly less than g follows immediately, by Corollary 5.6, and the reducible case in full generality will likewise follow once we have proved the irreducible case for genus g. To prove this case, we consider specifically a curve X_0 consisting of one component Y having genus 1 and no imposed ramification points, and the other component Z of genus $g-1$, with all imposed ramification points. By case (ii) of Lemma 5.7 and by our induction hypothesis, we have the dimensional upper bound on each component for any ramification sequences at the node, and non-emptiness when we consider the vanishing sequences $d - r - 1, d - r, \ldots, d - 2, d$ on Y, and $0, 2, 3, \ldots, r + 1$ on Z, so we obtain both statements for $G_d^r(X_0)$ by Corollary 5.6. We place X_0 in a smoothing family X/B [7, Thm. 3.4] with smooth generic fiber, and because the corresponding space $G_d^r(X)$ of relative limit series is proper, we conclude the dimensional upper bound and non-emptiness statements for the generic fiber, which is enough to imply them for a general curve, since $\mathcal{M}_{g,n}$ is connected.

Finally, in the case $\rho > 0$, we show connectedness. Fulton and Lazarsfeld [4] proved connectedness in the irreducible case. If we start with a reducible curve X_0 satisfying the hypotheses of our theorem, we place it as before in a smoothing family X/B with smooth generic fiber, and regular one-dimensional base. By Fulton-Lazarsfeld, the space of linear series is connected over the open subset of the base corresponding to smooth curves. Because X_0 is general, $G_d^r(X_0)$ has dimension ρ, and no component of $G_d^r(X)$ is supported over the special fiber. Because $G_d^r(X)$ is proper, we conclude also connectedness of the special fiber by, e.g., [9, Prop. 15.5.3]. □

Remark 5.8. In fact, we can use the same sort of arguments to give a sharp statement in terms of Schubert calculus on when G_d^r spaces are non-empty. However, this statement is already known [2, Rem. following Thm. 4.5], and we do not pursue it.

We conclude with a simple lemma in the limit linear series context. This lemma serves both to show that the upper bound of Corollary 5.5 is not sharp, and also

to show that in the case of "codimension 1" crude limit series, we still have a set-theoretic equivalence between limit series and Eisenbud-Harris limit series.

Lemma 5.9. *Continuing with the notation of Corollary 5.6, let (V^Y, V^Z) be a pair of linear series on Y and Z, each of degree d and dimension r, satisfying the desired ramification conditions, and such that for some j_0, we have*

$$d + 1 \geq a_{j_0}^{\Delta',Y} + a_{r-j_0}^{\Delta',Z} \geq d,$$

while for other j we still have $a_j^{\Delta',Y} + a_{r-j}^{\Delta',Z} = d$. Then there is a unique limit linear series on X restricting to (V^Y, V^Z).

Proof. In the case that

$$a_j^{\Delta',Y} + a_{r-j}^{\Delta',Z} = d$$

for all j, the arguments of [7, §6] come out of the following observation: this equality means that for any i, if we denote by d_i^Y the dimension of the space of sections of V^Y vanishing to order at least i at Δ', and d_i^Z the same for Z, then we have

$$d_i^Y + d_i^Z = \begin{cases} r + 1 : \nexists j \text{ such that } i = a_j^{\Delta',Y} = d - a_{r-j}^{\Delta',Z} \\ r + 2 : \text{ otherwise.} \end{cases}$$

For i such that $i \neq a_j^{\Delta',Y}$ for any j, all these sections vanish at Δ' on both Y and Z, and we can glue them arbitrarily in $\mathscr{L}^i$, so we get an $(r+1)$-dimensional space of possible sections of $\mathscr{L}^i$, and we are forced to choose this for V_i. If $i = a_j^{\Delta',Y}$ for some j, in principal we are choosing a V_i contained in an $(r+2)$-dimensional space, but here we have sections which are non-vanishing at Δ' on both Y and Z, so the requirement that they glue together at Δ' is a codimension 1 condition, and once again we are choosing V_i from an $(r+1)$-dimensional space.

But now suppose that we also allow

$$a_{j_0}^{\Delta',Y} + a_{r-j_0}^{\Delta',Z} = d + 1$$

for a certain j_0. The basic observation is that for the above argument to work, it suffices to know that $d_i^Y + d_i^Z \leq r + 2$, with equality only when the appropriate space of sections on either Y or Z has sections non-vanishing at Δ', since we still obtain a codimension 1 gluing condition in this case. But this is equivalent to having $d_i^Y + d_i^Z = r + 2$ only when $i = a_j^{\Delta',Y}$ or $i = d - a_{r-j}^{\Delta',Z}$ for some j, which one easily sees will follow from our hypothesis. $\square$

We can thus conclude:

Corollary 5.10. *Continuing with the hypotheses and notation of Theorem 1.1, if $\rho \leq 1$ we have that the natural set-theoretic map $G_d^r(X) \to G_d^{r,\mathrm{EH}}(X)$ is a bijection.*

Proof. We know by Theorem 5.3 that the map is always surjective, so it is enough to prove injectivity. Let (V^Y, V^Z) be a point of $G_d^{r,\mathrm{EH}}(X)$, with vanishing sequences a_i^Y and a_i^Z at the node. Because everything is general, for (V^Y, V^Z) to exist, by Theorem 1.1 we must have that ρ_Y and ρ_Z are both non-negative, after taking into account the ramification at the node. Furthermore, by the additivity of the Brill-Noether number we have that

$$\rho - \rho_Y - \rho_Z = \sum_j (a_j^Y + a_{r-j}^Z - d).$$

Since $\rho \leq 1$ and ρ_Y, ρ_Z, and the right hand side are all at least 0, we see that the right-hand side is either 0 or 1. The case that it is 0 is the case that (V^Y, V^Z) is refined, in which case we already knew that there is a unique point of $G_d^r(X)$ above it, and the case that it is 1 is the case addressed by the lemma. $\qquad\square$

REFERENCES

[1] David Eisenbud and Joe Harris, *Divisors on general curves and cuspidal rational curves*, Inventiones Mathematicae **74** (1983), 371–418.
[2] _____ , *Limit linear series: Basic theory*, Inventiones Mathematicae **85** (1986), 337–371.
[3] William Fulton, *Intersection theory*, second ed., Springer-Verlag, 1998.
[4] William Fulton and Robert Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, Acta Mathematica **146** (1981), 271–283.
[5] David Helm and Brian Osserman, *Flatness of the linked Grassmannian*, preprint.
[6] Deepak Khosla, *Moduli spaces of curves with linear series and the slope conjecture*, in preparation.
[7] Brian Osserman, *A limit linear series moduli scheme*, Annales de l'Institut Fourier, arXiv:math.AG/0407496, to appear.
[8] _____ , *Deformations of covers, Brill-Noether theory, and wild ramification*, Mathematical Research Letters **12** (2005), no. 4, 483–491.
[9] Alexander Grothendieck with Jean Dieudonné, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, troisiéme partie*, vol. 28, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1966.